

Robust Facility Location Under Demand Uncertainty and Facility Disruptions

Chun Cheng^{a*}, Yossiri Adulyasak^b, Louis-Martin Rousseau^c

^a*Institute of Supply Chain Analytics, Dongbei University of Finance and Economics, Dalian, 116025, Liaoning, China*

^b*HEC Montréal and GERAD, Montréal, H3T 2A7, Canada*

^c*Polytechnique Montréal and CIRRELT, Montréal, H3C 3A7, Canada*

* *Corresponding author. Email: chun.cheng@polymtl.ca*

Abstract

Facility location decision is strategic: the construction of a new facility is typically costly and the impact of the decision is long-lasting. Environmental changes, such as population shift and natural disasters, may cause today's optimal location decision to perform poorly in the future. Thus, it is important to consider potential uncertainties in the design phase, while explicitly taking into account the possible customer reassignments as recourse decisions in the execution phase. This paper studies a robust fixed-charge location problem under uncertain demand and facility disruptions. To model this problem, we adopt a two-stage robust optimization framework, where the first-stage location decision is made *here-and-now* and the second-stage allocation decision can be deferred until the uncertainty information is revealed. We develop a column-and-constraint generation (C&CG) algorithm to solve the models exactly and benchmark it with the other C&CG algorithm in the literature. We further extend our modeling and solution schemes to facility fortification problems under uncertainties, where investment decisions are made for already existing supply chain systems to protect facilities from disruptions and against uncertain demand. We conduct extensive numerical tests to study the differences in solutions produced by the three robust models and the impacts of uncertainties on solution configuration. Results show that our C&CG algorithm can solve more instances to optimality and consume less computing time on average, compared to the benchmark algorithm. Several managerial insights are also drawn from our numerical experiments. *Keywords:* Facility location, uncertain demand, disruption risk, robust optimization

1. Introduction

Facility location is an important aspect of strategic planning for both private companies and public sectors. Whether a manufacturer building a new plant, or a city planner choosing locations for public facilities, planners are often challenged by difficult resource allocation decisions (Owen and Daskin 1998).

The construction and acquisition of a new facility is typically a costly and time-consuming process. Therefore, once a new facility is built, it is expected to remain in operation for several years. However, environmental changes, such as population shift and transportation infrastructure issues, may turn today’s optimal location decision into tomorrow’s poor performance. It is therefore critical to consider potential uncertainties at the planning stage, to avoid high recourse costs at the operational stage.

In facility location problems, the uncertainty can be generally classified into three types: provider-side uncertainty, receiver-side uncertainty, and in-between uncertainty (Shen et al. 2011). The provider-side uncertainty includes uncertain supply capacity, lead time, and facility status (operational or failed). The receiver-side uncertainty captures randomness in demand. The in-between uncertainty involves uncertain transportation costs/times and arc status. These three types of uncertainty have been widely considered in the literature. For example, facility location under demand uncertainty (Atamtürk and Zhang 2007; Baron et al. 2011; Gülpınar et al. 2013), facility location under random travel costs/times (Nikoofal and Sadjadi 2010; Gao and Qin 2016; Mišković et al. 2017), and facility location under disruption risks (Cui et al. 2010; Yu et al. 2017; Afify et al. 2019).

Most studies consider one type of uncertainty at a time whereas a few of them consider simultaneous uncertainties, which are common in real-world applications of facility location where facilities are faced with multiple uncertainties at the same time. For example, at the time of building a new facility, it is typically very challenging to precisely predict the future demand of multiple customers over long-term horizons. In addition, during the operational stage, a facility could be disrupted by various risks such as power outages and natural disasters. A recent example is the supply chain disruptions caused by the coronavirus pandemic, where increased demand for personal protective equipment and medical devices occurred concurrently with reduced supplies due to shutdowns of manufacturing plants and service facilities (Besson 2020). Thus, in this work, we study a capacitated fixed-charge location problem (CFLP) that considers provider-side and receiver-side uncertainties simultaneously. More specifically, customers are subject to demand uncertainty and facilities may experience disruption risks. In practice, the exact distribution of random parameters is often unknown, or is estimated from limited historical data, which may result in severe inaccuracy in the solutions obtained (Roos and den Hertog 2020). Moreover, facility disruptions, especially those caused by natural disasters, are very difficult or impossible to predict. Thus, we adopt a two-stage robust optimization (RO) scheme to model the problem, which does not rely on any probabilistic information and is also less conservative in comparison with the static RO method. To solve the two-stage robust models exactly, we apply a column-and-constraint generation (C&CG) algorithm

based on a decomposition scheme. We benchmark this method with the other C&CG algorithm proposed by Simchi-Levi et al. (2019) for the two-stage robust network flow problem, a general problem that often arises from various supply chain applications.

Our contributions. We consider this study to make the following contributions: (1) To the best of our knowledge, this work is the first one to study the CFLP with simultaneous provider-side and receiver-side uncertainties in a two-stage RO framework. The corresponding model generalizes the problems with only demand uncertainty and with only facility disruptions. (2) We implement the C&CG method proposed by Zeng and Zhao (2013) and demonstrate empirically that it can solve the adjustable robust models efficiently in a reasonable runtime. Moreover, it shows better performance than the one proposed by Simchi-Levi et al. (2019). We further extend our modeling and solution schemes to facility fortification problems with uncertainties, where facility fortification decisions are made for already existing supply chain systems to protect facilities from disruptions and against demand uncertainty. (3) We conduct extensive numerical tests to study the differences in solutions produced by the three robust models, and the impacts of uncertainties on solution configuration and algorithm efficiency. Managerial insights are also drawn from numerical tests.

The rest of this paper is organized as follows. Section 2 reviews related literature. Section 3 presents the deterministic and the robust models. Section 4 describes the solution method and extends the proposed modeling and solution schemes to facility fortification problems under uncertainties. Section 5 discusses the numerical results. Section 6 concludes the paper.

2. Literature review

Flexible supply chains are expected to adapt effectively to supply disruptions and demand changes while maintaining customer service levels (Stevenson and Spring 2007). To mitigate supply chain risks, several strategies can be utilized to add redundancy to the system, e.g., inventory management, sourcing flexibility, demand substitution, and facility location (Snyder et al. 2016; Rajagopal et al. 2017). In this work, we use facility location to enhance supply chain flexibility, and thus related works on facility location under uncertainties are reviewed. A summary of the papers is given in Table 1.

For early studies of facility location under demand uncertainty, see the review paper by Snyder (2006). Baron et al. (2011) explore a multi-period facility location problem under demand uncertainty, where a box uncertainty set and an ellipsoid uncertainty set are used. Atamtürk and Zhang (2007) are the first to study the two-stage robust location-transportation problem (LTP), and a cutting plane algorithm

is applied. They compare solutions generated by the two-stage RO method, one-stage (also known as static) RO method, and the stochastic optimization method. For the same problem, Zeng and Zhao (2013) focus on comparing the performance of the C&CG algorithm and the Benders-style cutting plane method. Ardestani-Jaafari and Delage (2017) study a multi-period LTP and develop various approximation schemes to solve the problem based on the affine policy. Marandi and Van Houtum (2020) study a multi-period LTP with multiple items and integer-valued demand uncertainty. Adjustable RO formulations are developed for the problem. Basciftci et al. (2019) explore a facility location problem with decision-dependent uncertain demand, where location decisions affect the distribution of the underlying demand uncertainty. A distributionally robust optimization (DRO) model is constructed for the problem.

The in-between uncertainty is also studied in the literature. Gao and Qin (2016) study a p -hub center location problem under uncertain travel times. A chance constrained programming (CCP) approach is used and the deterministic equivalent model is solved by a genetic algorithm. Mišković et al. (2017) consider a two-echelon facility location problem, where products are first delivered to depots and then from depots to customers. The transportation costs in both echelons are uncertain. They use budgeted uncertainty sets and a static RO method for the problem. Matthews et al. (2019) address a single-commodity flow network design problem with multiple concurrent edge failures. They formulate the problem as a two-stage RO model and solve it with a C&CG algorithm.

In terms of facility disruptions, Yu et al. (2017) study the uncapacitated fixed-charge location problem (UFLP) in a stochastic optimization framework by incorporating risk preferences. They propose conditional value-at-risk- and absolute semideviation- based models to control the risk of transportation cost at each customer. Afify et al. (2019) study a reliable p -median problem (PMP) and a reliable UFLP, where each facility has a heterogeneous failure probability and each customer is allocated to a primary facility and a back-up facility. They propose an evolutionary learning algorithm to solve the problem. Xie et al. (2019) study the reliable UFLP with correlated disruptions, which are solved by Lagrangian relaxation based algorithms. Azad and Hassini (2019) explore the reliable supply chain network design problem with partial disruptions. A two-stage stochastic programming model is developed, which is solved by a Benders decomposition method. An et al. (2014) study a reliable PMP and Cheng et al. (2018b) consider a three-echelon logistics network design problem. Both studies use the two-stage RO framework and solve models using C&CG algorithms. For more details on reliable facility location, see the review paper by Snyder et al. (2016).

There also exist studies that consider multiple types of uncertainties simultaneously. For example,

Pishvae et al. (2011) study a closed-loop supply chain network design problem, where box uncertainty sets are used to describe the randomness in demand, returns, and transportation costs. Noyan et al. (2016) study a network design problem in the post-disaster environment, where demand- and network-related uncertainties are present. The authors use a two-stage stochastic programming method to model the problem, and develop a Benders decomposition-based branch-and-cut algorithm to solve it. Zetina et al. (2017) consider both uncertain demand and transportation costs in uncapacitated hub location problems. They use budgeted uncertainty sets to characterize uncertainties and the duality technique to reformulate the static robust models. For the same types of uncertainties, Wang et al. (2020) apply the DRO method to both the uncapacitated and capacitated hub location problems. Baghalian et al. (2013) study a supply chain network design problem, considering supply-side and demand-side uncertainties simultaneously. They assume demand variables follow a known distribution function and describe supply-side uncertainty (facility disruptions and link failures) through scenarios. Zokae et al. (2016) study a humanitarian network design problem, which includes suppliers, relief distribution centers, and affected areas. Installation costs of distribution centers, shortage costs at affected areas, transportation costs at both echelons, supply capacity, and demand are subject to uncertainties. They use budgeted uncertainty sets to describe randomnesses and a static RO method to model the problem. Mazahir and Ardestani-Jaafari (2020) consider a facility location problem in global sourcing under arc disruptions (resulting from supply-side uncertainty) and uncertain demand, which is solved by a two-stage RO scheme.

Table 1: A summary of related works

Authors	Type of uncertainty			Modeling scheme		Solution method
	Provider	Receiver	In-between	Static	Two(multi)-stage	
Baron et al. (2011)		✓		RO		Duality technique
Atamtürk and Zhang (2007)		✓			RO	Cutting plane
Zeng and Zhao (2013)		✓			RO	Column-and-constraint generation
Ardestani-Jaafari and Delage (2017)		✓			RO	Affine policy
Marandi and Van Houtum (2020)		✓			RO	Simplex-type method
Basciftci et al. (2019)		✓		DRO		Reformulation
Gao and Qin (2016)			✓	CCP		Genetic algorithm
Mišković et al. (2017)			✓	RO		Memetic algorithm
Matthews et al. (2019)			✓		RO	Column-and-constraint generation
Yu et al. (2017)	✓			Stochastic		Branch-and-cut, Lagrangian decomposition
Afify et al. (2019)	✓			Stochastic		Evolutionary learning algorithm
Xie et al. (2019)	✓			Stochastic		Lagrangian relaxation
Azad and Hassini (2019)	✓				Stochastic	Benders decomposition
An et al. (2014)	✓				RO	Column-and-constraint generation
Cheng et al. (2018b)	✓				RO	Column-and-constraint generation
Pishvae et al. (2011)		✓	✓	RO		Duality technique
Noyan et al. (2016)		✓	✓		Stochastic	Benders decomposition-based branch-and-cut
Zetina et al. (2017)		✓	✓	RO		Duality technique
Wang et al. (2020)		✓	✓	DRO		Reformulation
Baghalian et al. (2013)	✓	✓		Stochastic		Mixed-integer linear programming
Zokae et al. (2016)	✓	✓	✓	RO		Duality technique
Mazahir and Ardestani-Jaafari (2020)		✓	✓		RO	Column-and-constraint generation, affine policy
This work	✓	✓			RO	Column-and-constraint generation

From the literature, we can see that most works consider one type of uncertainty at a time. Although

some papers consider multiple types of uncertainties simultaneously, their modeling schemes may produce overly conservative solutions as all the decisions are made *here-and-now* (Pishvae et al. 2011; Zokae et al. 2016), or relatively optimistic solutions because it is impossible to enumerate all the disruption scenarios (Baghalian et al. 2013; Noyan et al. 2016). Thus, this paper uses a two-stage RO method for the CFLP under uncertain demand and facility disruptions, which utilizes revealed uncertainty information to make recourse decisions, to generate less conservative solutions. Moreover, the two-stage RO method does not depend on probability distribution or the generation of scenarios.

3. Models

This section presents the notation, the deterministic, and the robust models.

Notation. We denote \mathbb{R} as the space of real numbers and \mathbb{R}_+ as the space of positive real numbers. $|I|$ is the cardinality of set I . Let I and J be the sets of customers and facilities, respectively. The parameter f_j is the fixed cost of locating a facility at candidate site $j \in J$, and C_j is the corresponding capacity if we build a facility there. The parameter h_i is the demand quantity at customer $i \in I$, and d_{ij} is the transportation cost for facility j to satisfy one unit of demand at customer $i \in I$. The unit penalty cost associated with unmet demand at customer $i \in I$ is p_i . We use a binary variable $y_j = 1$ to denote that a facility is built at site $j \in J$, and $y_j = 0$ otherwise. The continuous variable x_{ij} is the product quantity delivered from facility $j \in J$ to customer $i \in I$, and variable u_i is the unsatisfied demand at customer $i \in I$.

3.1. The deterministic model

The deterministic CFLP can be formulated as follows:

$$\text{CFLP: } \min_{\mathbf{y}, \mathbf{x}, \mathbf{u}} \sum_{j \in J} f_j y_j + \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{i \in I} p_i u_i \quad (1a)$$

$$\text{s.t. } \sum_{j \in J} x_{ij} + u_i \geq h_i \quad \forall i \in I, \quad (1b)$$

$$\sum_{i \in I} x_{ij} \leq C_j y_j \quad \forall j \in J, \quad (1c)$$

$$y_j \in \{0, 1\} \quad \forall j \in J, \quad (1d)$$

$$x_{ij} \geq 0 \quad \forall i \in I, j \in J, \quad (1e)$$

$$u_i \geq 0 \quad \forall i \in I. \quad (1f)$$

The objective function (1a) minimizes the total cost, which includes the facility location cost, the transportation cost, and the penalty cost of unsatisfied demand. Constraints (1b) denote that the sum of met and unmet demand must be greater than or equal to a customer's demand. Constraints (1c) impose that customers can only be allocated to opened facilities and that a facility's capacity constraint must be respected. Constraints (1d)–(1f) impose the integrality and non-negativity constraints.

3.2. The adjustable robust model under uncertain demand and facility disruptions

Uncertainty set. We use a budgeted uncertainty set to characterize uncertain demand (Zeng and Zhao 2013; Bertsimas and Shtern 2018):

$$\mathcal{U}_h = \left\{ \mathbf{h} \in \mathbb{R}_+^{|I|} : h_i = \bar{h}_i + \theta_i h_i^\Delta, 0 \leq \theta_i \leq 1, \sum_{i \in I} \theta_i \leq \Gamma_h \right\}, \quad (2)$$

where \bar{h}_i is the nominal (or basic) demand at customer $i \in I$ and $h_i^\Delta \geq 0$ is the maximal demand deviation. Γ_h is the uncertainty budget which bounds the maximal number of demand parameters by which values are allowed to deviate from their nominal values.

We characterize disruption risks as (An et al. 2014; Cheng et al. 2018b)

$$\mathcal{Z}_k = \left\{ \mathbf{z} \in \{0, 1\}^{|J|} : \sum_{j \in J} z_j \leq k \right\}, \quad (3)$$

where $z_j = 1$ if facility $j \in J$ is disrupted, and $z_j = 0$ otherwise. Equation (3) means that at most k facilities are allowed to fail simultaneously in a disruption scenario.

We use the following uncertainty set to represent simultaneous demand uncertainty and facility disruptions

$$\mathcal{W} = \left\{ (\mathbf{h}, \mathbf{z}) \in \mathbb{R}_+^{|I|} \times \{0, 1\}^{|J|} : \mathbf{h} \in \mathcal{U}_h, \mathbf{z} \in \mathcal{Z}_k \right\}. \quad (4)$$

Adjustable robust model. The adjustable robust counterpart (ARC) model for the CFLP is

$$\text{CFLP-DR: } \min_{\mathbf{y}} \sum_{j \in J} f_j y_j + \max_{(\mathbf{h}, \mathbf{z}) \in \mathcal{W}} g(\mathbf{y}, \mathbf{h}, \mathbf{z}) \quad (5a)$$

$$\text{s.t. } y_j \in \{0, 1\} \quad \forall j \in J, \quad (5b)$$

where $g(\mathbf{y}, \mathbf{h}, \mathbf{z})$ is the optimal value of the second-stage problem defined as

$$\min_{\mathbf{x}, \mathbf{u}} \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{i \in I} p_i u_i \quad (6a)$$

$$\text{s.t. } \sum_{j \in J} x_{ij} + u_i \geq h_i \quad \forall i \in I, \quad (6b)$$

$$\sum_{i \in I} x_{ij} \leq C_j y_j (1 - z_j) \quad \forall j \in J, \quad (6c)$$

$$x_{ij} \geq 0 \quad \forall i \in I, j \in J, \quad (6d)$$

$$u_i \geq 0 \quad \forall i \in I. \quad (6e)$$

The objective function (5a) minimizes the sum of the facility location cost and the worst-case allocation cost. The *max* operator identifies the worst-case scenario and the *min* operator in problem (6) finds the least costly recourse solution (\mathbf{x}, \mathbf{u}) with a given first-stage location decision and a specific scenario. Constraints (6c) mean that customers can only be reassigned to opened and functional facilities (i.e., those with $y_j = 1$ and $z_j = 0$). We use CFLP-D and CFLP-R to denote the model with only uncertain demand and with only facility disruptions, respectively. The two models can be obtained directly by setting the parameter $\Gamma_h = 0$ or $k = 0$. From model (6), we can observe that uncertainties only affect the right-hand side of constraints and that the model has the property of *fixed recourse* (i.e., the coefficients of the recourse variables are not influenced by uncertainties).

Remark. *Random variables $\mathbf{z} \in \{0, 1\}^{|J|}$ can be relaxed to continuous variables, i.e., $0 \leq z_j \leq 1, j \in J$, to characterize facilities' partial disruptions, where a facility can still provide part of services to customers if it is opened and not completely disrupted. Under partial disruptions, the uncertainty budget k in set (3) can take any non-negative real value, which gives more flexibility to control the conservativeness of the robust solutions, compared to the case with k taking an integer value. More precisely, when \mathbf{z} are binary variables, k can still be set to a non-negative real value, but the worst-case scenario is always reached when the uncertainty budget is an integer value, i.e., at $\lfloor k \rfloor$.*

4. Solution method

In this section, we first introduce the solution method, and then extend our modeling and solution schemes to facility fortification models.

4.1. Column-and-constraint generation algorithm

In the two-stage RO framework, x_{ij} and u_i are no longer a single variable but rather a mapping from the space of observations $\mathbb{R}_+^{|I|} \times \{0, 1\}^{|J|}$ to $\mathbb{R}_+ \cup \{0\}$. This flexibility comes at the price of significant computational challenges. To solve the robust model, we may consider a vertex enumeration method, which reformulates the original robust model to a single mixed integer linear program (MILP) model by enumerating all the extreme points of the uncertainty set. However, in general the number of extreme vertices of a convex polyhedron is exponential in size with respect to the faces that describe it, which prevents us from using a full enumeration method. Therefore, we resort to the C&CG algorithm, which identifies a subset of the vertices of the uncertainty set and then applies a *reduced* vertex enumeration method. We use the CFLP-DR to describe the algorithm.

Our C&CG algorithm is based on the one developed by Zeng and Zhao (2013), which is implemented in a master-subproblem framework. The master problem (MP) is solved to generate the first-stage solution, and the subproblem (SP) is solved to identify the worst-case realization of the uncertain parameters under a given first-stage solution. Each time after an SP is solved, we compute the gap between the upper and lower bounds. If the optimality gap is reached, the algorithm terminates; otherwise, we add the identified worst-case scenario and its associated variables and constraints to the MP, and the algorithm iterates.

The master problem is written as

$$\phi = \min_{\mathbf{y}, s, \{\mathbf{x}^l\}_{l=1}^n, \{\mathbf{u}^l\}_{l=1}^n} s \quad (7a)$$

$$\text{s.t. } s \geq \sum_{j \in J} f_j y_j + \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij}^l + \sum_{i \in I} p_i u_i^l \quad \forall l \in \{1, \dots, n\}, \quad (7b)$$

$$\sum_{j \in J} x_{ij}^l + u_i^l \geq h_i^l \quad \forall l \in \{1, \dots, n\}, i \in I, \quad (7c)$$

$$\sum_{i \in I} x_{ij}^l \leq C_j y_j (1 - z_j^l) \quad \forall l \in \{1, \dots, n\}, j \in J, \quad (7d)$$

$$y_j \in \{0, 1\} \quad \forall j \in J, \quad (7e)$$

$$x_{ij}^l \geq 0 \quad \forall l \in \{1, \dots, n\}, i \in I, j \in J, \quad (7f)$$

$$u_i^l \geq 0 \quad \forall l \in \{1, \dots, n\}, i \in I. \quad (7g)$$

The MP seeks to find the best location decision in light of the set of worst-case scenarios identified in the subproblem. The allocation variables, x_{ij}^l and u_i^l , now feature an extra index l , which means that these variables are associated with the l th scenario (added after finishing the l th iteration). Similarly,

parameters h_i^l and z_j^l are the worst-case realizations of random variables h_i and z_j identified in the l th iteration via solving the subproblem. Since constraints (7b) are based on a subset of the uncertainty set \mathcal{W} , model (7) naturally provides a valid relaxation (or a lower bound) to the original two-stage RO model. By adding significant scenarios gradually to model (7), stronger lower bounds can be expected (Zeng and Zhao 2013).

To identify the significant scenarios, we solve the $\max_{(\mathbf{h}, \mathbf{z}) \in \mathcal{W}} g(\hat{\mathbf{y}}, \mathbf{h}, \mathbf{z})$ problem after getting a location decision $\hat{\mathbf{y}} \in \mathbb{R}^{|J|}$ from the MP. Since unmet demand is associated with a penalty cost, the second-stage problem is always feasible (a property termed as *relatively complete recourse*). Meanwhile, its optimal value is finite as the uncertainty set is bounded and nonempty. Thus, strong duality holds and we can use the Karush–Kuhn–Tucker (KKT) condition to derive the SP. Let $\boldsymbol{\alpha} \in \mathbb{R}^{|I|}$ and $\boldsymbol{\beta} \in \mathbb{R}^{|J|}$ be the dual variables associated with constraints (6b) and (6c), respectively, then the SP is written as

$$\begin{aligned}
\psi = & \max_{\substack{\mathbf{x}, \mathbf{u}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}^\alpha, \\ \mathbf{w}^\beta, \mathbf{w}^x, \mathbf{w}^u, \mathbf{z}, \boldsymbol{\theta}}} \sum_{j \in J} f_j \hat{y}_j + \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{i \in I} p_i u_i \\
\text{s.t. } & \sum_{j \in J} x_{ij} + u_i \geq \bar{h}_i + \theta_i h_i^\Delta & \forall i \in I, \\
& \sum_{i \in I} x_{ij} \leq C_j \hat{y}_j (1 - z_j) & \forall j \in J, \\
& \alpha_i - \beta_j \leq d_{ij} & \forall i \in I, j \in J, \\
& \alpha_i \leq p_i & \forall i \in I, \\
& \sum_{j \in J} x_{ij} + u_i \leq \bar{h}_i + \theta_i h_i^\Delta + M_i^\alpha (1 - w_i^\alpha) & \forall i \in I, \\
& \alpha_i \leq M_i^\alpha w_i^\alpha & \forall i \in I, \\
& \sum_{i \in I} x_{ij} \geq C_j \hat{y}_j (1 - z_j) + M_j^\beta (w_j^\beta - 1) & \forall j \in J, \\
& \beta_j \leq M_j^\beta w_j^\beta & \forall j \in J, \\
& \alpha_i - \beta_j \geq d_{ij} + M_{ij}^x (w_{ij}^x - 1) & \forall i \in I, j \in J \\
& x_{ij} \leq M_{ij}^x w_{ij}^x & \forall i \in I, j \in J \\
& \alpha_i \geq p_i + M_i^u (w_i^u - 1) & \forall i \in I, \\
& u_i \leq M_i^u w_i^u & \forall i \in I, \\
& \theta_i \leq 1 & \forall i \in I,
\end{aligned}$$

$$\sum_{i \in I} \theta_i \leq \Gamma_h,$$

$$\sum_{j \in J} z_j \leq k,$$

$$x_{ij}, u_i, \alpha_i, \beta_j, \theta_i \geq 0 \quad \forall i \in I, j \in J,$$

$$w_i^\alpha, w_j^\beta, w_{ij}^x, w_i^u, z_j \in \{0, 1\} \quad \forall i \in I, j \in J.$$

We set $M_i^\alpha = p_i$, $M_j^\beta = \max\{C_j, \max_i\{d_{ij}(\bar{h}_i + h_i^\Delta), p_i(\bar{h}_i + h_i^\Delta)\}\}$, $M_{ij}^x = \max\{C_j, d_{ij}(\bar{h}_i + h_i^\Delta), p_i(\bar{h}_i + h_i^\Delta)\} + d_{ij}$, $M_i^u = \max\{p_i, \bar{h}_i + h_i^\Delta\}$.

The detailed implementation of the C&CG algorithm is given in Algorithm 1. In the initialization step, we solve the deterministic model and get an initial location decision $\hat{\mathbf{y}}$. Note that the initial solution can be any feasible solution and not necessarily need to be that of the deterministic model. In the consequent steps, SP and MP are alternately solved to close the optimality gap. Specifically, in Step 1, we solve the SP with given $\hat{\mathbf{y}}$ to identify the worst-case scenario and update the upper bound. If the termination condition is satisfied (Step 2), the algorithm ends, else the identified worst-case scenario and its associated variables and constraints are added to the MP. In Step 3, the MP is solved and the iteration continues. The C&CG algorithm is guaranteed to converge in a finite number of iterations, which is upper bounded by the number of extreme points of the uncertainty set.

Algorithm 1: C&CG algorithm for the adjustable robust counterpart models

Initialization: Let $LB = -\infty$, $UB = \infty$, $n = 0$. Solve the deterministic model with $h_i = \bar{h}_i$ to get an initial location decision $\hat{\mathbf{y}}$. Set LB as the objective value of the deterministic model.

Iterate until the algorithm terminates:

Step 1: Solve the SP with given $\hat{\mathbf{y}}$ to find the worst-case scenario $(\hat{\mathbf{h}}, \hat{\mathbf{z}})$. Let $\hat{\psi}$ be the optimal value of the SP. Set $UB = \min\{UB, \hat{\psi}\}$ and $n = n + 1$.

Step 2: If $(UB - LB)/UB \leq \epsilon$, the algorithm terminates; else, add the identified worst-case scenario and its associated variables and constraints to the MP.

Step 3: Solve the MP to get a location decision $\hat{\mathbf{y}}$ and its optimal value $\hat{\phi}$. Set $LB = \hat{\phi}$ and go to Step 1.

Simchi-Levi et al. (2019) propose a new C&CG algorithm, which formulates the second-stage problem as a minimum cost flow problem. We provide the details of the C&CG algorithm developed by Simchi-Levi et al. (2019) in Appendix B. The main difference between our algorithm and theirs lies in the formulation of the subproblem (the master problems are the same). We use the KKT condition to derive the subproblem, which introduces big- M values to the model; whereas Simchi-Levi et al. (2019) reformulate the subproblem as a network flow problem, which introduces auxiliary binary variables to linearize the nonlinear terms in

the objective function. Table 2 presents the number of variables and constraints in the subproblems for the CFLP-DR, where P is a constant dependent on the maximum flow cost (i.e., $p^{max} = \max_{i \in I, j \in J} \{d_{ij}, p_i\}$) in the system, in particular, $P = \lceil \log_2(p^{max} + 1) \rceil - 1$. Table 2 shows that the size of our subproblem is decided by the number of nodes whereas that in Simchi-Levi et al. (2019) is affected by both the number of nodes and the maximum flow cost.

Table 2: Comparison of subproblems for the CFLP-DR

Authors	#Continuous variables	#Integer variables	#Constraints
Simchi-Levi et al. (2019)	$3 I + 2 J + 2 I P$	$2 I + 2 J + 3 + (I + 1)P$	$13 I + 10 J + 6 + I J + 8 I P$
This work	$3 I + J + I J $	$2 I + 2 J + I J $	$7 I + 3 J + 3 I J + 2$

Another common solution method for adjustable RO models is the affine policy, or affine decision rule, which restricts adjustable variables to be affine functions of uncertain parameters (Ben-Tal et al. 2004). This restriction is expected to produce a possibly conservative, yet tractable, robust counterpart. We also implemented this method for our problems, but results showed that in most cases, especially for the CFLP-D, the affine policy actually consumed more computational time than the C&CG algorithm. Thus, we choose to omit this method here. We also note that some decomposition methods are available for the robust counterpart obtained from the affine policy, and interested readers can refer to Ardestani-Jaafari and Delage (2017) and Ardestani-Jaafari and Delage (2019).

4.2. Application to facility fortification problems

In the previous sections, we suggest that we could improve the reliability of a system by considering disruption risks at the initial design phase. However, in reality, we may face situations where facilities are already existing and decision-makers are prone to improve the reliability of facilities by investing in protection and security measures, such as purchasing insurances or installing structural reinforcements, instead of constructing a system from scratch (Snyder et al. 2006; Liberatore et al. 2012; Qin et al. 2013; Tang et al. 2016). In this section, we discuss the extension of our modeling and solution schemes to facility fortification problems.

We build the model based on the notation in Section 3, with the exception that set J now denotes the set of already existing facilities. Correspondingly, parameter f_j is the fixed cost of fortifying facility $j \in J$. Variable $y_j = 1$ if facility $j \in J$ is chosen to be fortified, $y_j = 0$ otherwise. As in the four mentioned papers, we assume that a fortified facility becomes immune to disruptions. This assumption is widely used in the context of reliable facility location problems, where a reliable (or protected) facility is assumed to be nonfailable (Lim et al. 2010; Liberatore et al. 2011; Li et al. 2013; Qin et al. 2013; Zheng and Albert 2018;

Afify et al. 2019). The decision-maker has a total cost budget B for implementing fortification strategies. We also incorporate demand uncertainty to the fortification model, as at the moment of making facility enhancement decisions, future customer demand is often not known perfectly.

The two-stage robust facility fortification problem under uncertainties can be formulated as

$$\min_{\mathbf{y}} \max_{(\mathbf{h}, \mathbf{z}) \in \mathcal{W}} g'(\mathbf{y}, \mathbf{h}, \mathbf{z}) \quad (8a)$$

$$\text{s.t. } \sum_{j \in J} f_j y_j \leq B, \quad (8b)$$

$$y_j \in \{0, 1\} \quad \forall j \in J, \quad (8c)$$

where $g'(\mathbf{y}, \mathbf{h}, \mathbf{z})$ is defined by Equations (6a)–(6b), (6d)–(6e), and

$$\sum_{i \in I} x_{ij} \leq C_j - C_j(1 - y_j)z_j \quad \forall j \in J. \quad (9)$$

Constraints (9) mean that a fortified facility (i.e., $y_j = 1$) is always available with capacity C_j . If facility j is not fortified, it can still supply C_j units of product when $z_j = 0$, as the facility is already existing; however, it cannot serve any customer when $z_j = 1$. Note that besides imposing a budget constraint on the fortification cost, we can also optimize the cost term $\sum_{j \in J} f_j y_j$ as in Section 3, i.e., we include it in the objective function and omit the budget constraint.

Correspondingly, we re-write the master problem of the C&CG algorithm as

$$\phi = \min_{\mathbf{y}, s, \{\mathbf{x}^l\}_{l=1}^n, \{\mathbf{u}^l\}_{l=1}^n} s \quad (10a)$$

$$\text{s.t. } s \geq \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij}^l + \sum_{i \in I} p_i u_i^l \quad \forall l \in \{1, \dots, n\} \quad (10b)$$

$$\sum_{i \in I} x_{ij}^l \leq C_j - C_j(1 - y_j)z_j^l \quad \forall l \in \{1, \dots, n\}, j \in J. \quad (10c)$$

$$\text{and (7c), (7f)–(7g), (8b)–(8c).} \quad (10d)$$

Constraints (10c) suggest that a fortified facility is always operational, even if it might be identified to be disrupted in some worst-case scenarios.

The subproblem is augmented through replacing the objective function and the two capacity-related

constraints by

$$\psi = \max_{\substack{\mathbf{x}, \mathbf{u}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}^\alpha, \\ \mathbf{w}^\beta, \mathbf{w}^x, \mathbf{w}^u, \mathbf{z}}} \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{i \in I} p_i u_i \quad (11a)$$

$$\sum_{i \in I} x_{ij} \leq C_j - C_j(1 - \hat{y}_j)z_j \quad \forall j \in J, \quad (11b)$$

$$\sum_{i \in I} x_{ij} \geq C_j - C_j(1 - \hat{y}_j)z_j + M_j^\beta(\omega_j^\beta - 1) \quad \forall j \in J. \quad (11c)$$

In terms of computational complexity, the master problem (10) has one more constraint, i.e., the budget constraint (8b), compared to the master problem in Section 4.1. And the number of constraints of the subproblem keeps unchanged. Meanwhile, no additional variables are introduced to the algorithm. Therefore, we can also expect the C&CG algorithm to solve the robust fortification model efficiently.

5. Computational experiments

For our numerical tests, we adopt the instance sets in Cheng et al. (2018a) with slight modifications, which are based on real-world geographic data set of the United States—the 49-node case presented in Daskin (2011). These instances are derived from 1990 US census data. The 49 nodes include the state capitals of the continental United States and Washington, D.C. The same data set has been widely used in the context of reliable facility location, e.g., Snyder and Daskin (2005), Cui et al. (2010), Lu et al. (2015), and Azad and Hassini (2019). For our problem, there are 35 instances in total. The nominal demand $\bar{h}_i = P_i \times 10^{-5}$, where P_i is the population at node $i \in I$. We generate the maximal demand deviation h_i^Δ uniformly from the interval $[0.15\bar{h}_i, \bar{h}_i]$. The transportation cost d_{ij} is the great circle distance between node $i \in I$ and node $j \in J$ in miles. For simplicity, we set the unit penalty cost p_i , $i \in I$, the same for all the customers, which is the greatest travel distance in the system. We denote instances as *Fac-X-Cus-Y*, which means that the considered instance has X candidate facilities and Y customers.

All the algorithms and models were coded in Python programming language, using Gurobi 8.1.1 as the solver. The calculations were run on a cluster of Lenovo SD350 servers with 2.4 GHz Intel Skylake cores and 202 GB of memory under Linux CentOS 7 system. Each experiment was conducted on a four-core processor of one node.

5.1. The impact of uncertainty on optimal solutions

We use instances Fac-10-Cus-10 and Fac-15-Cus-15 to conduct the experiments and set $\Gamma_h = 0.2|I|$ and $k = 2$. Results are presented in Table 3. We set the deterministic model's results as benchmarks and

the other models' results are normalized by dividing those of the deterministic model. A ratio smaller (or larger) than 1 means that the robust models generate solutions of smaller (or larger) costs. Note that the nominal costs of the robust models are calculated by fixing the location decisions and solving the resulting minimum cost flow problems. The worst-case cost of the deterministic model is obtained by fixing the location decision and solving the subproblem of the C&CG algorithm with the same uncertainty parameters as the robust models.

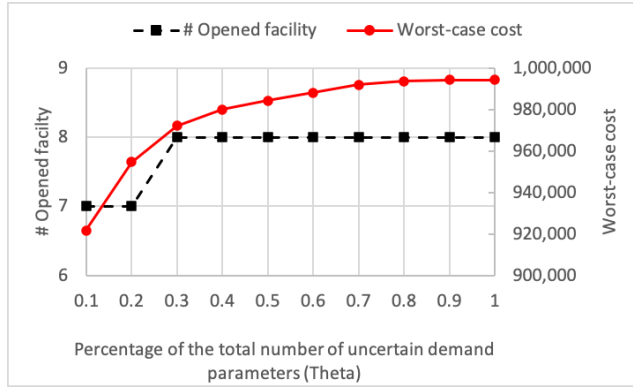
Table 3: The impact of uncertainty on location decision and cost

Instance	Model	Opened facilities	Nominal cost ratio	Worst-case cost ratio
Fac-10-Cus-10	CFLP	[0*, 2, 3, 6, 8]	1.00	1.00
	CFLP-D	[0, 2, 3, 5, 6, 8]	1.04	0.94
	CFLP-R	[0, 2, 3, 4, 5, 6, 8]	1.08	0.68
	CFLP-DR	[0, 2, 3, 4, 5, 6, 7, 8]	1.13	0.73
Fac-15-Cus-15	CFLP	[0, 1, 2, 3, 4, 5, 7]	1.00	1.00
	CFLP-D	[0, 1, 2, 3, 4, 5, 7]	1.00	1.00
	CFLP-R	[0, 1, 2, 3, 4, 5, 7, 14]	1.09	0.94
	CFLP-DR	[0, 1, 2, 3, 4, 5, 7, 10, 14]	1.17	0.82

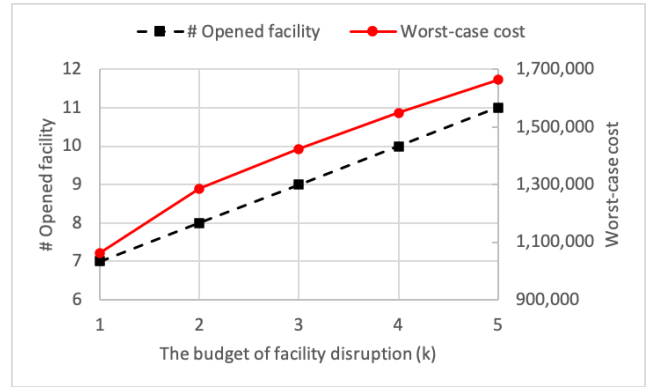
* Facilities are indexed from 0.

The first impact is the selection of opened facilities. As expected, when uncertainties are considered, more facilities are opened to mitigate potential risks. The CFLP-DR model generates solutions of the greatest number of opened facilities, due to the fact that two types of uncertainties are considered simultaneously. The second impact is cost. Generally, considering uncertainty increases the nominal cost and decreases the worst-case cost. Moreover, the nominal cost of model CFLP-DR has the greatest increase in comparison with the deterministic model. A similar phenomenon is also observed by Zetina et al. (2017), where the system has the largest cost when two types of uncertainties are simultaneously considered, compared to one type of uncertainty.

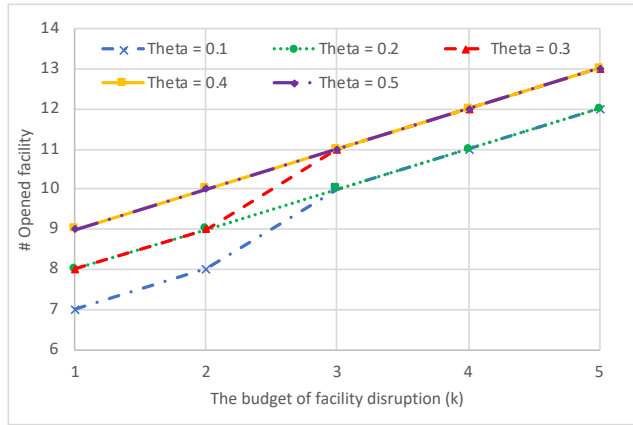
Table 3 also shows that facility disruption risk (or provider-side uncertainty) has a greater influence on the location decisions and the costs, compared to demand uncertainty. For instance Fac-10-Cus-10, the CFLP-D opens one more facility than the deterministic model. However, when disruption risk is further considered, the CFLP-DR locates two more facilities compared to the CFLP-D. For instance Fac-15-Cus-15, the location decision of the CFLP-D is the same as the deterministic problem. However, the CFLP-R and CFLP-DR generate different solutions with more opened facilities. Mazahir and Ardestani-Jaafari (2020) also indicate that the supplier's uncertainty is a more pressing issue than the effects of demand uncertainty. Their computational results show that 3% provider-side uncertainty has almost the same impact on the worst-case profit as 30% demand uncertainty. Table 3 further displays that sometimes a



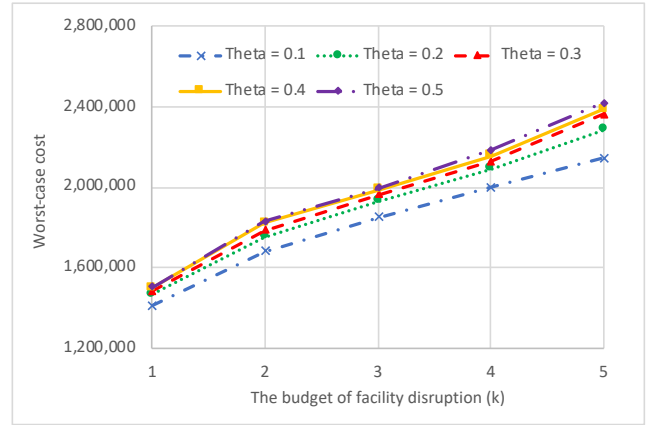
(a) CFLP-D



(b) CFLP-R



(c) CFLP-DR



(d) CFLP-DR

Figure 1: The impact of uncertainty budget on optimal solutions (Instance Fac-15-Cus-15)

slight increase in the nominal cost can lead to a significant decrease in the worst-case cost. For example, for instance Fac-10-Cus-10, the nominal cost ratio of the CFLP-R (CFLP-DR) is 1.08 (1.13) whereas the worst-case cost ratio is 0.68 (0.73). This observation is consistent with other works that study facility location problems under disruptions (Snyder and Daskin 2005; An et al. 2014).

5.2. The impact of uncertainty budget on optimal solutions

We denote $\Gamma_h = \Theta|I|$, which means there are at most Θ of customers whose demand parameters are allowed to deviate from their nominal values. For space consideration, we only present the results of instance Fac-15-Cust-15 in Figure 1.

Figure 1(a) indicates that the budget of demand uncertainty has a slight impact on the location decision and the worst-case cost after a threshold. Specifically, when Θ increases from 0.2 to 0.3, one more facility is opened. When $\Theta \geq 0.3$, the number of opened facilities stays the same, and the rate of the cost growth is relatively small. The system does not open more facilities when the demand uncertainty increases, because it is more cost-efficient to simply penalize the unmet demand instead of paying the fixed

expenses for locating new facilities. On the contrary, the budget of facility disruption has a significant influence on facility configuration and the worst-case cost. Figure 1(b) displays that the number of opened facilities and the worst-case cost increase quickly with respect to k for the CFLP-R. Cheng et al. (2018b) also notice that the worst-case cost increases almost linearly over the uncertainty budget in a three-echelon network design problem under disruptions.

Figures 1(c)–1(d) are the summarized results of the CFLP-DR. The detailed results are given in Table 4, where the last column is obtained by setting the results in the first row of different Θ values as benchmarks. Both figures show that the uncertainty of facility disruptions plays a dominating role. Under the same value of k , the maximal difference in the number of opened facilities is 2, and the worst-case costs are also close. However, under the same value of Θ , the number of opened facilities and the worst-case cost ratio increase almost linearly with k . We note that sometimes even the number of opened facilities is the same under the *same* value of k for *different* Θ , there might be a difference in facility configuration. For example, according to Table 4, when $k = 4$, 11 sites are opened when $\Theta = 0.1$ and $\Theta = 0.2$. However, facilities 6 and 8 are opened in the former circumstance, and facilities 1 and 11 are opened in the latter. Thus, to improve supply chain flexibility via location decisions, we can either increase the number of opened facilities or change the facility configuration under a fixed number of opened facilities. For example, in the robust p -median problem (An et al. 2014), where exactly p facilities should be opened under all the situations, a selected set of p facilities to be opened in the first stage can differ when the uncertainty budget changes.

5.3. Analyses of algorithm performance

This section first compares our C&CG algorithm with that proposed by Simchi-Levi et al. (2019), and then explores the impact of uncertainty budget on algorithm performance. The optimality tolerance for algorithms is set to 0.01% and the CPU time limit is set to 7200 seconds. Note that, when this limit is reached, we still allow the current iteration to be finished unless the walltime limit–10800 seconds–is reached. In following tables, $\#Opt$ is the number of instances out of 35 that are solved to optimality. $\#Iter$ is the number of iterations. Gap is the optimality gap between the upper and lower bounds.

5.3.1. Performance comparison

We compare the algorithms by setting $\Gamma_h = 0.2|I|$ and $k = 2$. Experiments are conducted on model CFLP-DR where both types of uncertainties are considered. In order to use the C&CG algorithm in Simchi-Levi et al. (2019) for this problem, the assumption that all the transportation costs $d_{ij}, i \in I, j \in J$

Table 4: Detailed results of the CFLP-DR for Instance Fac-15-Cust-15

Θ	k	Opened facilities	#Opened facilities	Worst-case cost ratio
0.1	1	[0, 1, 2, 3, 4, 5, 7]	7	1.00
	2	[0, 1, 2, 4, 5, 7, 10, 14]	8	1.19
	3	[0, 1, 2, 4, 5, 7, 10, 11, 13, 14]	10	1.31
	4	[0, 2, 3, 4, 5, 6 , 7, 8 , 10, 13, 14]	11	1.41
	5	[0, 1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14]	12	1.52
0.2	1	[0, 1, 2, 3, 4, 5, 7, 14]	8	1.00
	2	[0, 1, 2, 3, 4, 5, 7, 10, 14]	9	1.20
	3	[0, 1, 2, 4, 5, 7, 10, 11, 13, 14]	10	1.32
	4	[0, 1 , 2, 3, 4, 5, 7, 10, 11 , 13, 14]	11	1.42
	5	[0, 1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14]	12	1.56
0.3	1	[0, 1, 2, 3, 4, 5, 7, 14]	8	1.00
	2	[0, 1, 2, 3, 4, 5, 7, 10, 14]	9	1.20
	3	[0, 2, 3, 4, 5, 7, 8, 10, 11 , 13, 14]	11	1.32
	4	[0, 2, 3, 4, 5, 6, 7, 8 , 10, 11, 13, 14]	12	1.43
	5	[0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14]	13	1.59
0.4	1	[0, 1, 2, 3, 4, 5, 7, 11, 14]	9	1.00
	2	[0, 1, 2, 3, 4, 5, 7, 10, 11, 14]	10	1.21
	3	[0, 1 , 2, 3, 4, 5, 7, 8, 10, 13, 14]	11	1.32
	4	[0, 1 , 2, 3, 4, 5, 6, 7, 10, 11, 13, 14]	12	1.44
	5	[0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14]	13	1.59
0.5	1	[0, 1, 2, 3, 4, 5, 7, 11, 14]	9	1.00
	2	[0, 1, 2, 3, 4, 5, 7, 10, 11, 14]	10	1.22
	3	[0, 1 , 2, 3, 4, 5, 7, 8, 10, 13, 14]	11	1.32
	4	[0, 1 , 2, 3, 4, 5, 6, 7, 10, 11, 13, 14]	12	1.45
	5	[0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14]	13	1.61

and penalty costs $p_i, i \in I$ are integer is required. Thus, we round them to the nearest integers in the experiments in this subsection. Results are reported in Table 5, where $\#Fac$ and $\#Cust$ are the number of facilities and customers, respectively. We mark the results with less CPU time and a smaller optimality gap in bold.

Table 5 displays that our C&CG algorithm can solve 33 out of 35 instances to optimality within the time limit. However, the C&CG algorithm proposed by Simchi-Levi et al. (2019) can only solve 18 instances and the maximum walltime limit is reached in 14 instances. For the 18 instances that are solved to optimality by both algorithms, our C&CG algorithm generally consumes less CPU time. We further observe that the time difference of the two algorithms mainly results from the resolution time used in the subproblem. As indicated by Table 2, the size of the subproblem in Simchi-Levi et al. (2019) is related to the value of parameter P . Thus, for realistic size problems with relatively high flow costs, the subproblem in Simchi-Levi et al. (2019) has a greater computational complexity in comparison with the subproblem derived from the KKT condition. Based on the results here, our C&CG algorithm is used for further experiments in the subsequent sections.

Table 5: Algorithm comparison on model CFLP-DR

#Fac	#Cust	Our C&CG				C&CG in Simchi-Levi et al. (2019)			
		CPU time (seconds)				CPU time (seconds)			
		Total	MP	SP	Gap	Total	MP	SP	Gap
10	10	2.91	0.23	2.68	0.00	5.57	0.23	5.34	0.00
	15	5.42	0.24	5.18	0.00	24.75	0.28	24.47	0.00
	20	28.69	0.51	28.18	0.00	42.48	0.53	41.95	0.00
	25	19.47	0.34	19.13	0.00	279.67	0.35	279.32	0.00
	30	59.99	0.49	59.50	0.00	116.03	0.57	115.46	0.00
	35	218.09	0.32	217.77	0.00	307.52	0.34	307.18	0.00
	40	189.74	0.12	189.62	0.00	79.70	0.13	79.57	0.00
	45	291.14	0.40	290.74	0.00	135.88	0.46	135.42	0.00
	49	355.35	0.50	354.85	0.00	431.46	0.58	430.88	0.00
15	15	23.86	2.23	21.63	0.00	27.62	2.21	25.41	0.00
	20	21.39	0.85	20.54	0.00	351.66	0.88	350.78	0.00
	25	79.17	5.86	73.31	0.00	1242.28	6.22	1236.06	0.00
	30	196.99	6.15	190.84	0.00	4211.47	6.41	4205.06	0.00
	35	84.79	1.39	83.40	0.00	2825.44	1.48	2823.96	0.00
	40	298.99	7.99	291.00	0.00	7981.79	4.07	7977.72	1.08
	45	435.26	3.29	431.97	0.00	T			
	49	560.31	5.22	555.09	0.00	T			
20	20	38.34	2.85	35.49	0.00	619.23	3.54	615.69	0.00
	25	290.60	31.07	259.53	0.00	2274.41	33.64	2240.77	0.00
	30	195.57	18.70	176.87	0.00	8895.59	20.13	8875.46	0.00
	35	299.15	18.51	280.64	0.00	8197.95	0.54	8197.41	30.83
	40	412.84	17.29	395.55	0.00	T			
	45	546.90	11.23	535.67	0.00	8550.16	0.78	8549.38	19.51
	49	645.02	33.20	611.82	0.00	T			
25	25	242.27	46.25	196.02	0.00	6409.23	47.87	6361.36	0.00
	30	404.90	80.88	324.02	0.00	T			
	35	456.49	131.57	324.92	0.00	T			
	40	709.49	132.95	576.54	0.00	T			
	45	2566.22	261.17	2305.05	0.00	T			
	49	3694.82	120.48	3574.34	0.00	T			
30	30	892.16	292.67	599.49	0.00	T			
	35	2075.79	912.55	1163.24	0.00	T			
	40	2083.45	977.17	1106.28	0.00	T			
	45	7309.41	3961.57	3347.84	1.54	T			
	49	7327.06	5432.46	1894.60	0.14	T			

T: the walltime limit is reached.

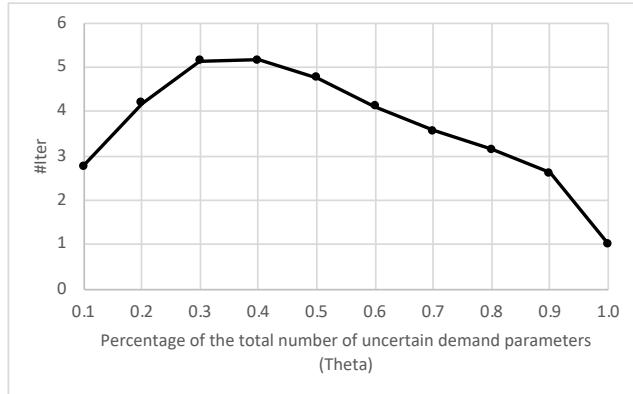
5.3.2. The impact of uncertainty budget on algorithm performance

This section studies the impact of uncertainty budget on algorithm performance. Experiments are conducted for different types of uncertainties as presented in Section 3.2 to provide computational insights.

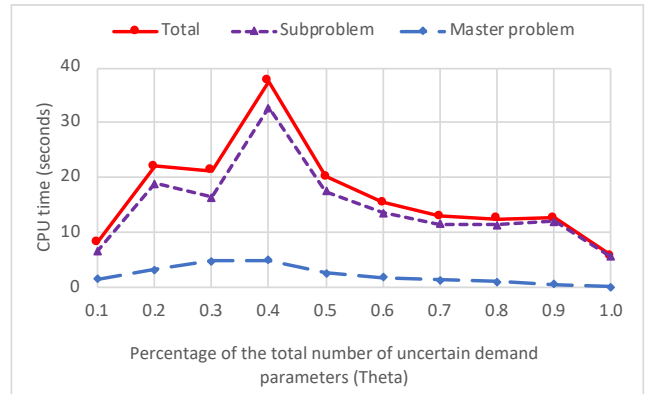
The CFLP-D. Table 6 and Figure 2 present the results of the CFLP-D. Table 6 shows that our C&CG algorithm can generate optimal solutions for all the instances with different budgets in a short time, and the number of iterations of the algorithm is also small. Figure 2(a) shows that the #Iter first increases and then decreases over the uncertainty budget. When $\Theta = 1$, the C&CG algorithm finds optimal solutions in only 1 iteration and identifies that all the uncertain parameters would take value 1. Figure 2(b) shows that for the CFLP-D, most time is consumed to solve the subproblem and the computing time of the master problem is relatively shorter.

Table 6: Algorithm performance for the CFLP-D over different budgets (average results)

Θ	#Opt	#Iter	Gap	CPU time (seconds)		
				Total	MP	SP
0.1	35/35	2.77	0.00	7.99	1.48	6.49
0.2	35/35	4.20	0.00	22.02	3.13	18.85
0.3	35/35	5.14	0.00	21.32	4.82	16.45
0.4	35/35	5.17	0.00	37.55	4.87	32.63
0.5	35/35	4.77	0.00	20.08	2.60	17.43
0.6	35/35	4.11	0.00	15.43	1.86	13.54
0.7	35/35	3.57	0.00	12.95	1.37	11.56
0.8	35/35	3.14	0.00	12.38	0.95	11.41
0.9	35/35	2.63	0.00	12.62	0.56	12.04
1.0	35/35	1.00	0.00	5.65	0.08	5.57



(a) #Iter



(b) CPU time

Figure 2: Average results of algorithm performance for the CFLP-D

The CFLP-R. Table 7 and Figure 3 present the results of the CFLP-R. Table 7 displays that the C&CG algorithm can solve all the instances to optimality for a small budget $k = 1$ and 2 in a short time framework. Although the #Opt decreases to 25 and 18 for $k = 3$ and 4, the average optimality gap is not

significant, i.e., 2.32% and 6.83% respectively. Figure 3 shows that the number of iterations and the total computing time increase over k . In contrast to the results of the CFLP-D, the master problem consumes more CPU time than the subproblem. This can be explained by the fact that as the number of iterations increases, the size of the master problem gradually increases as more variables and constraints are added to it.

Table 7: Algorithm performance for the CFLP-R over different budgets (average results)

k	#Opt	#Iter	Gap	CPU time (seconds)		
				Total	MP	SP
1	35/35	4.57	0.00	16.12	5.84	10.25
2	35/35	14.37	0.00	390.27	281.74	108.25
3	25/35	34.11	2.32	2660.07	2124.10	534.74
4	18/35	42.91	6.83	3970.05	3126.92	841.44

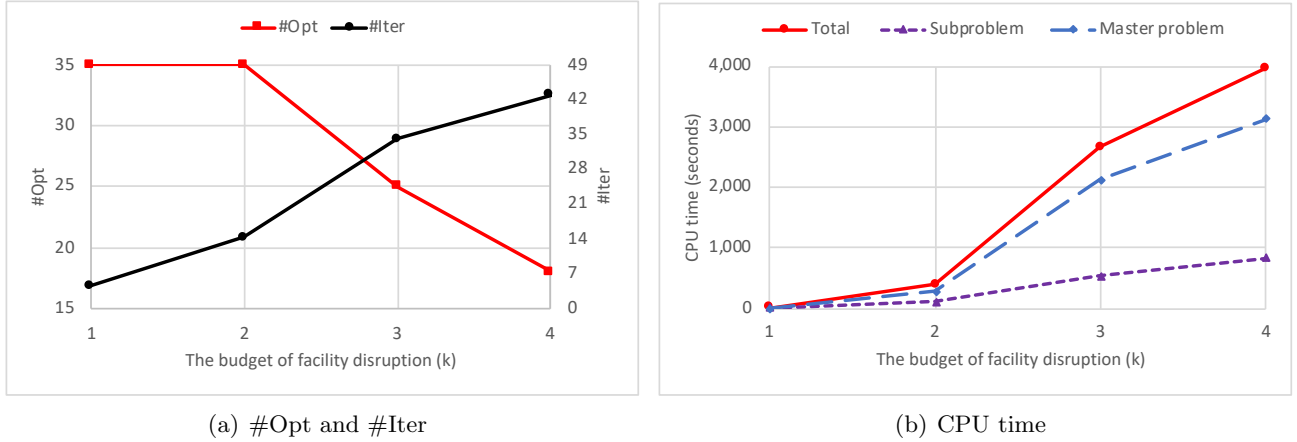


Figure 3: Average results of algorithm performance for the CFLP-R

The CFLP-DR. Table 8 summarizes the results of the CFLP-DR. It shows that when both uncertainties are simultaneously considered, fewer instances (33, 33, 34 out of 35) are solved to optimality for the case of $k = 2$, whereas all the instances are optimally solved for the CFLP-R when $k = 2$. When $k = 3$ and 4, the optimality gaps are close to those of the CFLP-R. However, we observe that the gaps increase over Θ under the same value of k in general. Figure 4(a) further shows that when $k = 4$, the #Opt has a relatively larger variation under different values of Θ , varying between 22 and 18. We also notice that under the same value of Θ , the #Opt has an even larger variation with respect to k , especially for the case of $\Theta = 0.6$ (#Opt varies between 35 and 18). Figure 4(b) displays that the case of $\Theta = 0.2$ has the maximal number of iterations in general, leading to a longer computing time for the master problem (as shown in Figure 4(d)); however, for the other two cases, their subproblems consume much more time, resulting in a longer total CPU time as reported in Figure 4(c). In addition, we can observe that the

disruption risk has a greater impact on computational efficiency than that of the demand uncertainty. In particular, under a fixed value of k , the variations of the #Opt and the total CPU time over Θ are not as obvious as those under a fixed value of Θ but with a varying k .

Table 8: Algorithm performance for the CFLP-DR over different budgets (average results)

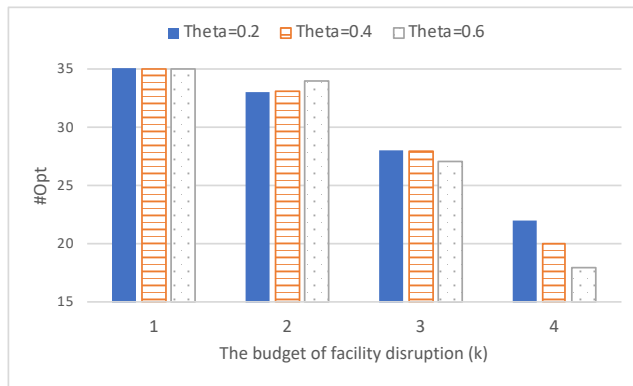
Θ	k	#Opt	#Iter	Gap	CPU time (seconds)		
					Total	MP	SP
0.2	1	35/35	8.00	0.00	163.50	27.31	136.06
	2	33/35	14.00	0.05	944.05	348.08	595.64
	3	28/35	24.63	1.38	2500.03	641.84	1857.46
	4	22/35	28.49	5.15	3494.25	940.69	2552.62
0.4	1	35/35	8.57	0.00	331.03	28.37	302.53
	2	33/35	14.69	0.04	1391.08	319.11	1071.62
	3	28/35	21.40	2.02	2915.23	387.11	2527.53
	4	20/35	22.29	7.24	3960.69	513.61	3446.44
0.6	1	35/35	8.14	0.00	325.37	12.51	312.75
	2	34/35	13.69	0.07	1344.43	170.79	1173.34
	3	27/35	18.00	2.74	3114.85	285.34	2829.04
	4	18/35	18.60	7.87	4372.40	316.42	4055.53

Summaries. (1) Among the three robust models, the CFLP-D is the easiest one in terms of computational complexity, for which all the instances can be solved to optimality in a short time framework. (2) For the CFLP-R, when the uncertainty budget is small, all the instances can be optimally solved; for a large budget, the average optimality gap is promising within the time limit. Meanwhile, the master problem consumes more time than the subproblem because of the great number of iterations. (3) The CFLP-DR is the most difficult one among the three robust models, which requires the most computational time. Zetina et al. (2017) also find that the robust counterpart under two types of uncertainties requires the longest computing time, compared to that under a single source of uncertainty. (4) The disruption risk has a larger impact on algorithm performance than the demand uncertainty.

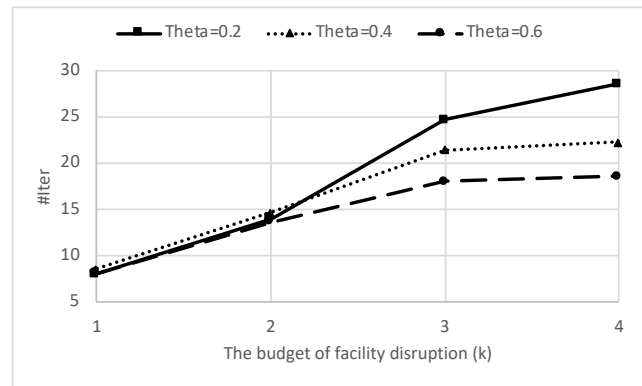
5.4. Insights from the robust fortification models

This section provides results for the robust fortification models. Experiments are conducted using a randomly selected instance Fac-10-Cus-30. For the models with both disruptions and uncertain demand, we set the budget of demand uncertainty $\Gamma_h = 0.3|I|$.

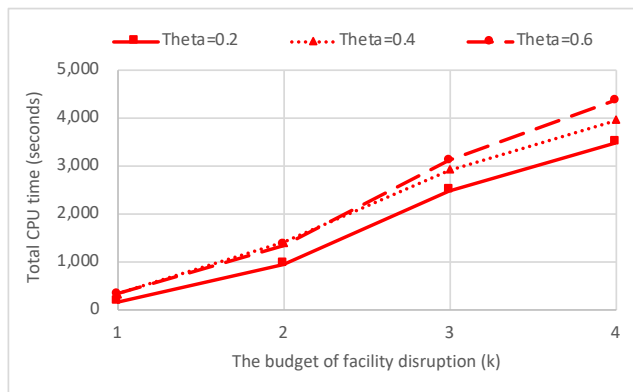
In the first group of experiments, we study the impact of budget B on the system's worst-case cost by setting $B = \sigma \sum_{j \in J} f_j$, where $\sigma \in \{10\%, \dots, 60\%\}$. Results are reported in Figure 5. It shows that the worst-case cost decreases as the cost budget increases because more investments lead to more flexibility in facility fortification decisions, such as increasing the number of fortified facilities or fortifying an expensive



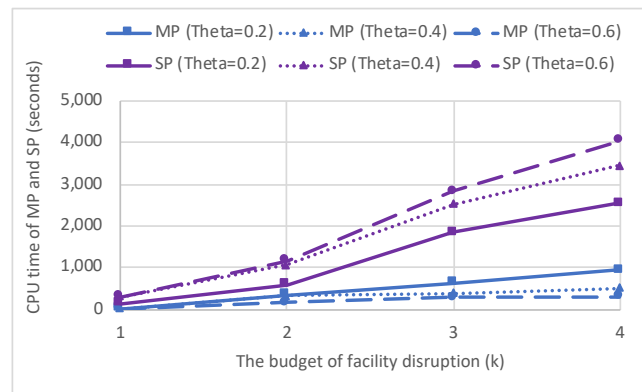
(a) #Opt



(b) #Iter

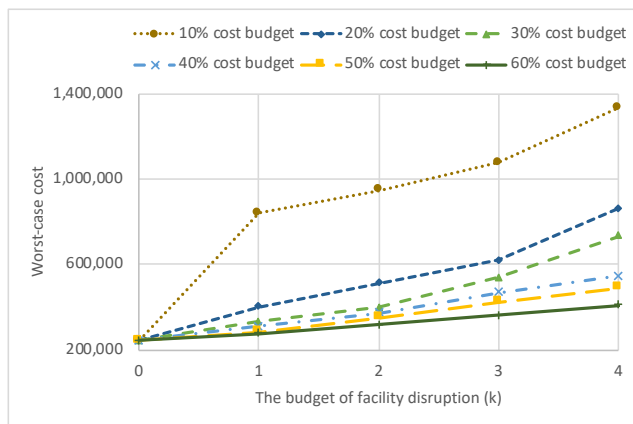


(c) Total CPU time

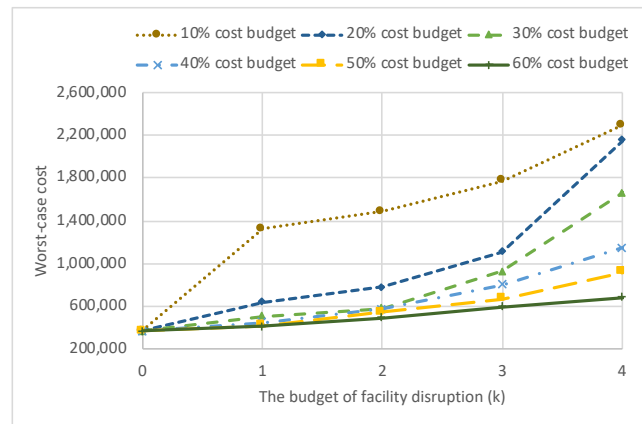


(d) CPU time of master and sub problems

Figure 4: Average results of different algorithms for the CFLP-DR



(a) Only facility disruptions



(b) Both facility disruptions and demand uncertainty

Figure 5: Worst-case cost under different cost budgets and uncertainty budgets (Instance Fac-10-Cus-30)

facility with a large capacity. When σ increases from 10% to 30%, the worst-case cost has a quick decrease under the same value of k , especially for the case with both types of uncertainties (as shown in Figure 5(b)). However, if we further increase the cost budget from 30% to 60%, the worst-case cost begins to decrease slowly, in particular when $k \leq 3$. Thus, the robust fortification models can be used as decision tools for managers to see the trade-off between the investment in protection measures and the resulting robustness of a system.

Table 9: Comparison between robust CFLP models and robust fortification models

CFLP-R				Fortification model with only facility disruptions		
k	Opened facilities	Facility cost	Worst-case flow Cost	Fortified facilities	Facility cost	Worst-case flow cost
1	[0, 2, 3, 4, 5, 7, 8]	478100	966320.64	[0, 1, 2, 3, 5]	421800	276697.39
2	[0, 2, 3, 4, 5, 6, 7, 8]	544100	1203400.07	[0, 1, 2, 3, 4, 5, 6]	526200	295728.79
3	[0, 2, 3, 4, 5, 6, 7, 8, 9]	640700	1284592.17	[0, 1, 2, 3, 4, 5, 6, 9]	622800	270611.69
4	[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]	742500	1510399.81	[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]	742500	242932.87
CFLP-DR				Fortification model with both types of uncertainties		
k	Opened facilities	Facility cost	Worst-case flow cost	Fortified facilities	Facility cost	Worst-case flow cost
1	[0, 2, 3, 4, 5, 6, 7, 8]	544100	1405946.47	[0, 1, 2, 3, 5, 9]	518400	407327.56
2	[0, 2, 3, 4, 5, 6, 7, 8, 9]	640700	1728833.74	[0, 1, 2, 3, 4, 5, 6, 9]	622800	403133.55
3	[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]	742500	1982132.83	[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]	742500	371110.91
4	[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]	742500	2546018.16	[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]	742500	371110.91

Next, we compare the robust fortification models with the robust CFLP models. To perform the experiments, we first solve the robust CFLP models and get the location decisions and costs. For the robust fortification models, we then set the budget B as the location cost of the corresponding robust CFLP model. This experimental setting is to explore the impacts of different strategies (obtained from different models) under the same cost budget on system robustness. Results are reported in Table 9. It shows that the location or fortification costs of the two types of models are very close; however, there is a significant gap in their worst-case flow costs. This is because in the fortification models, if a facility is chosen to be protected, it becomes immune to disruptions; whereas in the robust CFLP models, a new located facility can be disrupted later and thus loses its capacity completely. Decision-makers can then leverage these models based on their cost budget to make an optimal trade-off among different strategies to improve overall supply chain system’s reliability and resilience.

Based on the experiments in this section, we make the following summaries to help business managers and policymakers to improve their supply chain practices and facility reliability: (1) Considering uncertainties at the facility design phase can effectively reduce recourse costs at the operational phase; thus, when making strategic decisions, supply chain managers should proactively evaluate possible uncertain-

ties they may face during the tactic or/and operational stages, and then incorporate these uncertainties into the decision-making process, to better balance system reliability and costs in the long run. (2) The uncertainty budget, especially on the provider-side uncertainty, has a significant impact on the nominal and worst-case costs. Therefore, when implementing the RO frameworks, solutions based on different levels of uncertainty budget can be examined by the decision-maker to properly determine the facility configuration which yields an optimal trade-off for the firm. In addition, if data is available, this budget parameter can also be calibrated using statistical hypothesis tests (see Bertsimas et al. (2018)). (3) For already existing facilities, managers can use the adjustable robust fortification model to see the required investments in protection measures and resulting robustness. (4) The proposed two-stage RO scheme can be applied to solve various real-world location problems, e.g., the site-selection and capacity allocation problem for the municipal solid waste management system in the Tehran city of Iran (Habibi et al. 2017) and the location-inventory problem for mitigating disruption risk during the Atlantic hurricane season in the US (Velasquez et al. 2020). Besides industrial applications, the two-stage RO approach also applies to public facility location problems, e.g., the locations of healthcare facilities (Ahmadi-Javid et al. 2017), which are also constrained by resources and challenged by uncertainties.

6. Conclusions

This paper solves a fixed-charge location problem where two types of parameters are subject to uncertainties simultaneously: demand and facility availability. We apply a two-stage RO framework for the problem, which allows the allocation decisions to be made after the uncertainties are realized. We implement the C&CG method proposed by Zeng and Zhao (2013) to solve the robust models exactly, and further extend the modeling and solution schemes to facility fortification problems under uncertainties. We benchmark our C&CG algorithm with the one developed by Simchi-Levi et al. (2019), where the second-stage problem is reformulated to a minimum cost flow problem. Numerical tests show that our algorithm can solve more instances to optimality and generally outperform the benchmark approach. Results also indicate that, among the three robust models, the CFLP-D is the easiest one in terms of computational complexity, for which all the instances can be optimally solved under different budgets. The CFLP-DR is the most difficult one, as two types of uncertainties are simultaneously considered. Our tests further demonstrate that disruption risk (or provider-side uncertainty) has a greater effect on solution configuration and cost, compared to demand uncertainty. To summarize, our solution optimization framework allows decision-makers to determine an optimal and robust facility location decision to improve

overall reliability and resilience of the supply chain facing complex uncertainties.

Acknowledgements

The authors gratefully acknowledge the support of Institute for Data Valorization (IVADO), the Fonds de Recherche du Québec – Nature et Technologies (FRQNT), and the Natural Sciences and Engineering Research Council of Canada (Grant 2016-05822).

References

- Affy, B., Ray, S., Soeanu, A., Awasthi, A., Debbabi, M., and Allouche, M. (2019). Evolutionary learning algorithm for reliable facility location under disruption. *Expert Systems with Applications*, 115:223–244.
- Ahmadi-Javid, A., Seyedi, P., and Syam, S. S. (2017). A survey of healthcare facility location. *Computers & Operations Research*, 79:223–263.
- An, Y., Zeng, B., Zhang, Y., and Zhao, L. (2014). Reliable p -median facility location problem: Two-stage robust models and algorithms. *Transportation Research Part B: Methodological*, 64:54–72.
- Ardestani-Jaafari, A. and Delage, E. (2017). The value of flexibility in robust location–transportation problems. *Transportation Science*, 52(1):189–209.
- Ardestani-Jaafari, A. and Delage, E. (2019). Linearized robust counterparts of two-stage robust optimization problems with applications in operations management. *NFORMS Journal on Computing*. Forthcoming.
- Atamtürk, A. and Zhang, M. (2007). Two-stage robust network flow and design under demand uncertainty. *Operations Research*, 55(4):662–673.
- Azad, N. and Hassini, E. (2019). A benders decomposition method for designing reliable supply chain networks accounting for multimitigation strategies and demand losses. *Transportation Science*, 53(5):1287–1312.
- Baghalian, A., Rezapour, S., and Farahani, R. Z. (2013). Robust supply chain network design with service level against disruptions and demand uncertainties: A real-life case. *European Journal of Operational Research*, 227(1):199–215.
- Baron, O., Milner, J., and Naseraldin, H. (2011). Facility location: A robust optimization approach. *Production and Operations Management*, 20(5):772–785.
- Basciftci, B., Ahmed, S., and Shen, S. (2019). Distributionally robust facility location problem under decision-dependent stochastic demand. *arXiv preprint arXiv:1912.05577*.

- Ben-Tal, A., Goryashko, A., Guslitzer, E., and Nemirovski, A. (2004). Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376.
- Bertsimas, D., Gupta, V., and Kallus, N. (2018). Data-driven robust optimization. *Mathematical Programming*, 167(2):235–292.
- Bertsimas, D. and Shtern, S. (2018). A scalable algorithm for two-stage adaptive linear optimization. *arXiv preprint arXiv:1807.02812*.
- Besson, E. K. (2020). Covid-19 (coronavirus): Panic buying and its impact on global health supply chains. Accessed November 7, 2020, <https://blogs.worldbank.org/health/covid-19-coronavirus-panic-buying-and-its-impact-global-health-supply-chains>.
- Cheng, C., Adulyasak, Y., and Rousseau, L.-M. (2018a). Robust facility location under disruptions. *Technical report*. <https://www.gerad.ca/en/papers/G-2018-91/view>.
- Cheng, C., Qi, M., Zhang, Y., and Rousseau, L.-M. (2018b). A two-stage robust approach for the reliable logistics network design problem. *Transportation Research Part B: Methodological*, 111:185–202.
- Cui, T., Ouyang, Y., and Shen, Z.-J. M. (2010). Reliable facility location design under the risk of disruptions. *Operations Research*, 58(4-part-1):998–1011.
- Daskin, M. S. (2011). *Network and Discrete Location: Models, Algorithms, and Applications*. John Wiley & Sons.
- Gao, Y. and Qin, Z. (2016). A chance constrained programming approach for uncertain p -hub center location problem. *Computers & Industrial Engineering*, 102:10–20.
- Gülpınar, N., Pachamanova, D., and Çanaköğlü, E. (2013). Robust strategies for facility location under uncertainty. *European Journal of Operational Research*, 225(1):21–35.
- Gupte, A., Ahmed, S., Cheon, M. S., and Dey, S. (2013). Solving mixed integer bilinear problems using milp formulations. *SIAM Journal on Optimization*, 23(2):721–744.
- Habibi, F., Asadi, E., Sadjadi, S. J., and Barzinpour, F. (2017). A multi-objective robust optimization model for site-selection and capacity allocation of municipal solid waste facilities: A case study in tehran. *Journal of Cleaner Production*, 166:816–834.
- Li, Q., Zeng, B., and Savachkin, A. (2013). Reliable facility location design under disruptions. *Computers & Operations Research*, 40(4):901–909.
- Liberatore, F., Scaparra, M. P., and Daskin, M. S. (2011). Analysis of facility protection strategies against an uncertain number of attacks: The stochastic r -interdiction median problem with fortification. *Computers & Operations Research*, 38(1):357–366.

- Liberatore, F., Scaparra, M. P., and Daskin, M. S. (2012). Hedging against disruptions with ripple effects in location analysis. *Omega*, 40(1):21–30.
- Lim, M., Daskin, M. S., Bassamboo, A., and Chopra, S. (2010). A facility reliability problem: Formulation, properties, and algorithm. *Naval Research Logistics (NRL)*, 57(1):58–70.
- Lu, M., Ran, L., and Shen, Z.-J. M. (2015). Reliable facility location design under uncertain correlated disruptions. *Manufacturing & Service Operations Management*, 17(4):445–455.
- Marandi, A. and Van Houtum, G.-J. (2020). Robust location-transportation problems with integer-valued demand. *Optimization Online*.
- Matthews, L. R., Gounaris, C. E., and Kevrekidis, I. G. (2019). Designing networks with resiliency to edge failures using two-stage robust optimization. *European Journal of Operational Research*, 279(3):704–720.
- Mazahir, S. and Ardestani-Jaafari, A. (2020). Robust global sourcing under compliance legislation. *European Journal of Operational Research*, 284(1):152–163.
- McCormick, G. P. (1976). Computability of global solutions to factorable nonconvex programs: Part I—Convex underestimating problems. *Mathematical Programming*, 10(1):147–175.
- Mišković, S., Stanimirović, Z., and Grujičić, I. (2017). Solving the robust two-stage capacitated facility location problem with uncertain transportation costs. *Optimization Letters*, 11(6):1169–1184.
- Nikoofal, M. E. and Sadjadi, S. J. (2010). A robust optimization model for p -median problem with uncertain edge lengths. *The International Journal of Advanced Manufacturing Technology*, 50(1-4):391–397.
- Noyan, N., Balcik, B., and Atakan, S. (2016). A stochastic optimization model for designing last mile relief networks. *Transportation Science*, 50(3):1092–1113.
- Owen, S. H. and Daskin, M. S. (1998). Strategic facility location: A review. *European Journal of Operational Research*, 111(3):423–447.
- Pishvaei, M. S., Rabbani, M., and Torabi, S. A. (2011). A robust optimization approach to closed-loop supply chain network design under uncertainty. *Applied Mathematical Modelling*, 35(2):637–649.
- Qin, X., Liu, X., and Tang, L. (2013). A two-stage stochastic mixed-integer program for the capacitated logistics fortification planning under accidental disruptions. *Computers & Industrial Engineering*, 65(4):614–623.
- Rajagopal, V., Venkatesan, S. P., and Goh, M. (2017). Decision-making models for supply chain risk mitigation: A review. *Computers & Industrial Engineering*, 113:646–682.

- Roos, E. and den Hertog, D. (2020). Reducing conservatism in robust optimization. *INFORMS Journal on Computing*. Forthcoming.
- Shen, Z.-J. M., Zhan, R. L., and Zhang, J. (2011). The reliable facility location problem: Formulations, heuristics, and approximation algorithms. *INFORMS Journal on Computing*, 23(3):470–482.
- Simchi-Levi, D., Wang, H., and Wei, Y. (2019). Constraint generation for two-stage robust network flow problems. *INFORMS Journal on Optimization*, 1(1):49–70.
- Snyder, L. V. (2006). Facility location under uncertainty: A review. *IIE Transactions*, 38(7):547–564.
- Snyder, L. V., Atan, Z., Peng, P., Rong, Y., Schmitt, A. J., and Sinoysal, B. (2016). OR/MS models for supply chain disruptions: A review. *IIE Transactions*, 48(2):89–109.
- Snyder, L. V. and Daskin, M. S. (2005). Reliability models for facility location: The expected failure cost case. *Transportation Science*, 39(3):400–416.
- Snyder, L. V., Scaparra, M. P., Daskin, M. S., and Church, R. L. (2006). Planning for disruptions in supply chain networks. In *Models, methods, and applications for innovative decision making*, pages 234–257. INFORMS.
- Stevenson, M. and Spring, M. (2007). Flexibility from a supply chain perspective: definition and review. *International Journal of Operations & Production Management*, 27(7):685–713.
- Tang, L., Zhu, C., Lin, Z., Shi, J., and Zhang, W. (2016). Reliable facility location problem with facility protection. *PloS one*, 11(9):e0161532.
- Velasquez, G. A., Mayorga, M. E., and Özaltın, O. Y. (2020). Prepositioning disaster relief supplies using robust optimization. *IIE Transactions*, pages 1–19.
- Wang, S., Chen, Z., and Liu, T. (2020). Distributionally robust hub location. *Transportation Science*, 54(5):1189–1210.
- Xie, S., An, K., and Ouyang, Y. (2019). Planning facility location under generally correlated facility disruptions: Use of supporting stations and quasi-probabilities. *Transportation Research Part B: Methodological*, 122:115–139.
- Yu, G., Haskell, W. B., and Liu, Y. (2017). Resilient facility location against the risk of disruptions. *Transportation Research Part B: Methodological*, 104:82–105.
- Zeng, B. and Zhao, L. (2013). Solving two-stage robust optimization problems using a column-and-constraint generation method. *Operations Research Letters*, 41(5):457–461.
- Zetina, C. A., Contreras, I., Cordeau, J.-F., and Nikbakhsh, E. (2017). Robust uncapacitated hub location. *Transportation Research Part B: Methodological*, 106:393–410.

Zheng, K. and Albert, L. A. (2018). An exact algorithm for solving the bilevel facility interdiction and fortification problem. *Operations Research Letters*, 46(6):573–578.

Zokaee, S., Bozorgi-Amiri, A., and Sadjadi, S. J. (2016). A robust optimization model for humanitarian relief chain design under uncertainty. *Applied Mathematical Modelling*, 40(17-18):7996–8016.

Appendix A. The static robust optimization model

The static RO model for the CFLP with facility disruptions is

$$\begin{aligned}
& \min_{\mathbf{y}, \mathbf{x}, \mathbf{u}} \sum_{j \in J} f_j y_j + \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{i \in I} p_i u_i, \\
& \text{s.t.} \quad \sum_{j \in J} x_{ij} + u_i \geq \bar{h}_i && \forall i \in I, \\
& \quad \sum_{i \in I} x_{ij} \leq C_j y_j (1 - z_j) && \forall z \in \mathcal{Z}_k, j \in J, \\
& \quad y_j \in \{0, 1\} && \forall j \in J, \\
& \quad x_{ij} \geq 0 && \forall i \in I, j \in J, \\
& \quad u_i \geq 0 && \forall i \in I.
\end{aligned}$$

Through duality theory, the second constraints can be reformulated as

$$\sum_{i \in I} x_{ij} \leq C_j y_j - k A_j - B_j \quad \forall j \in J, \quad (\text{A.1})$$

$$C_j y_j - A_j - B_j \leq 0 \quad \forall j \in J, \quad (\text{A.2})$$

$$A_j, B_j \geq 0 \quad \forall j \in J. \quad (\text{A.3})$$

Constraints (A.2)–(A.3) indicate that when $k \geq 1$, the equation $C_j y_j - k A_j - B_j \leq 0$ always holds. Therefore, when $k \geq 1$, we have $\sum_{i \in I} x_{ij} \leq 0, j \in J$, which suggests that $x_{ij} = 0, i \in I, j \in J$ and $u_i = \bar{h}_i, i \in I$. Since the static RO model is a minimization problem, all y_j would be 0 at optimality. A logical explanation of this result is that when $k \geq 1$, the adversary can always select an opened facility to disrupt, thus rendering the problem infeasible in the absence of recourse.

Appendix B. C&CG algorithm of Simchi-Levi et al. (2019)

Simchi-Levi et al. (2019) reformulate the subproblem of the C&CG algorithm as a minimum cost flow problem instead of using the KKT condition as we do. To transform the second-stage problem to a minimum cost flow problem, we first add a dummy facility node indexed by $|J| + 1$ and a dummy demand node indexed by $|I| + 1$ to the supply chain system. For notational convenience, we denote $I' = I \cup \{|I| + 1\}$ and $J' = J \cup \{|J| + 1\}$. We then add arcs from site $|J| + 1$ to demand node $i \in I$ with transportation cost p_i , and arcs from $j \in J'$ to demand node $|I| + 1$ with zero transportation cost. We assume the supply capacity at site $|J| + 1$ is

$$C_{|J|+1} = \sum_{i \in I} h_i,$$

and the demand quantity at customer $|I| + 1$ is

$$h_{|I|+1} = \sum_{j \in J} C_j y_j (1 - z_j).$$

Then, the net flow balance condition always holds for the considered system. Note that facility $|J| + 1$ is always operational and its fixed cost is 0. Now we can equivalently formulate $g(\mathbf{y}, \mathbf{h}, \mathbf{z})$ as a minimum cost flow problem:

$$\min_{\mathbf{x}} \sum_{i \in I'} \sum_{j \in J'} d_{ij} x_{ij} \tag{B.1a}$$

$$\text{s.t.} \quad - \sum_{j \in J'} x_{ij} = -h_i \quad \forall i \in I', \tag{B.1b}$$

$$\sum_{i \in I'} x_{ij} = C_j y_j (1 - z_j) \quad \forall j \in J, \tag{B.1c}$$

$$\sum_{i \in I'} x_{i,|J|+1} = C_{|J|+1} \tag{B.1d}$$

$$x_{ij} \geq 0 \quad \forall i \in I', j \in J'. \tag{B.1e}$$

We derive the dual problem of model (B.1) as

$$\max_{\boldsymbol{\alpha}, \boldsymbol{\beta}} - \sum_{i \in I'} h_i \alpha_i + \sum_{j \in J} C_j y_j (1 - z_j) \beta_j + C_{|J|+1} \beta_{|J|+1} \tag{B.2a}$$

$$\text{s.t.} \quad -\alpha_i + \beta_j \leq d_{ij} \quad \forall i \in I', j \in J', \tag{B.2b}$$

$$\boldsymbol{\alpha} \in \mathbb{R}^{|I'|}, \boldsymbol{\beta} \in \mathbb{R}^{|J'|}. \tag{B.2c}$$

Based on Assumption 3 (i.e., the unit flow cost associated with each arc is nonnegative integer) in Simchi-Levi et al. (2019), we assume that transportation cost d_{ij} , $i \in I'$, $j \in J'$ are integer. This assumption is critical to the fact that we can further restrict α and β to be nonnegative integers based on Proposition 2 in Simchi-Levi et al. (2019). More specifically, if α and β are nonnegative integers, we can replace them with their binary representations as in Equation (B.5), and then linearize the nonlinear term in objective (B.4).

According to Proposition 2 in Simchi-Levi et al. (2019), model (B.2) is equivalent to

$$\max_{\alpha, \beta} - \sum_{i \in I'} h_i \alpha_i + \sum_{j \in J} C_j y_j (1 - z_j) \beta_j + C_{|J|+1} \beta_{|J|+1} \quad (\text{B.3a})$$

$$\text{s.t.} \quad -\alpha_i + \beta_j \leq d_{ij} \quad \forall i \in I', j \in J', \quad (\text{B.3b})$$

$$\alpha_i \leq \alpha^{max}, \beta_j \leq \beta^{max} \quad \forall i \in I', j \in J', \quad (\text{B.3c})$$

$$\alpha \in \mathbb{N}^{|I'|}, \beta \in \mathbb{N}^{|J'|}, \quad (\text{B.3d})$$

where $\alpha^{max} = \beta^{max} = \max_{i \in I', j \in J'} d_{ij}$ and \mathbb{N} denotes nonnegative integers. The objective function (B.3a) can be explicitly written as

$$\max \sum_{i \in I} \bar{h}_i (\beta_{|J|+1} - \alpha_i) + \sum_{j \in J} C_j y_j (\beta_j - \alpha_{|I|+1}) + \sum_{i \in I} h_i^\Delta \theta_i (\beta_{|J|+1} - \alpha_i) - \sum_{j \in J} C_j y_j z_j (\beta_j - \alpha_{|I|+1}), \quad (\text{B.4})$$

where the last two terms are nonlinear. The third term contains the product of continuous variables (i.e., θ_i) and integer variables (i.e., $\beta_{|J|+1}$ and α_i), whereas the fourth term involves the product of binary variables (i.e., z_j) and integer variables (i.e., β_j and $\alpha_{|I|+1}$). To linearize the third term, we replace integer variables $\beta_{|J|+1}$ and α_i , $i \in I$ with their binary representations (Gupte et al. 2013)

$$\beta_{|J|+1} = \sum_{k=0}^P 2^k \beta_{|J|+1}^k, \quad \alpha_i = \sum_{k=0}^P 2^k \alpha_i^k, \quad (\text{B.5})$$

where $\beta_{|J|+1}^k$, α_i^k , $k \in \{0, \dots, P\}$ are binary variables and $P = \left\lceil \log_2 \left(\max_{i \in I', j \in J'} d_{ij} + 1 \right) \right\rceil - 1$. Up to now, we can use the McCormick (1976) envelopes to linearize the third term of objective (B.4). Let

$W_{i,|J|+1}^k = \theta_i \beta_{|J|+1}^k$ and $Q_i^k = \theta_i \alpha_i^k$, then

$$\sum_{i \in I} h_i^\Delta \theta_i (\beta_{|J|+1} - \alpha_i) = \sum_{i \in I} h_i^\Delta \left(\sum_{k=0}^P 2^k (W_{i,|J|+1}^k - Q_i^k) \right) \quad (\text{B.6})$$

and

$$W_{i,|J|+1}^k \leq \beta_{|J|+1}^k, \quad W_{i,|J|+1}^k \leq \theta_i, \quad W_{i,|J|+1}^k \geq \theta_i - (1 - \beta_{|J|+1}^k), \quad W_{i,|J|+1}^k \geq 0, \quad (\text{B.7})$$

$$Q_i^k \leq \alpha_i^k, \quad Q_i^k \leq \theta_i, \quad Q_i^k \geq \theta_i - (1 - \alpha_i^k), \quad Q_i^k \geq 0. \quad (\text{B.8})$$

We can directly use the McCormick (1976) envelopes to linearize the fourth term of objective (B.4).

Let $T_j = z_j \beta_j$ and $U_j = z_j \alpha_{|I|+1}$, then

$$-\sum_{j \in J} C_j y_j z_j (\beta_j - \alpha_{|I|+1}) = -\sum_{j \in J} C_j y_j (T_j - U_j) \quad (\text{B.9})$$

and

$$T_j \leq \beta^{max} z_j, \quad T_j \leq \beta_j, \quad T_j \geq \beta_j - (1 - z_j) \beta^{max}, \quad T_j \geq 0, \quad (\text{B.10})$$

$$U_j \leq \alpha^{max} z_j, \quad U_j \leq \alpha_{|I|+1}, \quad U_j \geq \alpha_{|I|+1} - (1 - z_j) \alpha^{max}, \quad U_j \geq 0. \quad (\text{B.11})$$

Finally, we get the subproblem of the C&CG algorithm in Simchi-Levi et al. (2019) as follows

$$\begin{aligned} \max \quad & \sum_{j \in J} f_j \hat{y}_j + \sum_{i \in I} \bar{h}_i (\beta_{|J|+1} - \alpha_i) + \sum_{j \in J} C_j y_j (\beta_j - \alpha_{|I|+1}) + \\ & \sum_{i \in I} h_i^\Delta \left(\sum_{k=0}^P 2^k (W_{i,|J|+1}^k - Q_i^k) \right) - \sum_{j \in J} C_j y_j (T_j - U_j) \\ \text{s.t.} \quad & -\alpha_i + \beta_j \leq d_{ij} \quad \forall i \in I', j \in J', \\ & \alpha_i \leq \alpha^{max}, \quad \forall i \in I', \\ & \beta_j \leq \beta^{max} \quad \forall j \in J', \\ & \alpha \in \mathbb{N}^{|I'|}, \beta \in \mathbb{N}^{|J'|}, \\ & \beta_{|J|+1} = \sum_{k=0}^P 2^k \beta_{|J|+1}^k, \\ & \alpha_i = \sum_{k=0}^P 2^k \alpha_i^k, \quad \forall i \in I, \end{aligned}$$

$$\begin{aligned}
0 \leq \theta_i \leq 1 & \quad \forall i \in I, \\
\sum_{i \in I} \theta_i \leq \Gamma_h, \\
\sum_{j \in J} z_j \leq k, \\
z_j \in \{0, 1\}, & \quad \forall j \in J, \\
\beta_{|J|+1}^k, \alpha_i^k \in \{0, 1\} & \quad \forall i \in I, k \in \{0, \dots, P\}
\end{aligned}$$

(B.7)–(B.8) and (B.10)–(B.11).